Stability in Poisson Geometry

Abstract

In these lectures I will discuss the theory of stability of leaves in various geometric settings. The main focus will be in Poisson geometry. I will give a detailed proof of a fundamental result about the stability of symplectic leaves of Poisson manifolds. I will also discuss a "universal stability result" for Lie algebroids.

IST

November 1, 2008

Authored by: Rui Loja Fernandes

Stability in Poisson Geometry

Plan of the lectures

Lecture 1. Stability of leaves: overview [2,5,7]

Stability in dynamical systems: periodic orbits of vector fields. Stability in foliated geometry: stability of leaves of foliations; the theorem of Reeb, Thurston, Langevin & Rosenberg. Stability in equivariant geometry: the theorem of Hirsch and Stowe. The stability problem for symplectic leaves: stability *vs* strong stability.

Lecture 2. Basic concepts in Poisson Geometry [1]

Poisson brackets and Poisson tensors. Hamiltonian vector fields. Basic examples: symplectic manifolds, symplectic foliations, duals of Lie algebras, fiberwise linear Poisson structures. Basic constructions. The splitting theorem. The symplectic foliation. Fixed points and isotropy Lie algebra. The linearization problem: Conn's linearization theorem.

Lecture 3. Cartan Calculus in Poisson Geometry [1,4,6]

The cotangent Lie algebroid:anchor, Lie bracket, differential. Contraction, flow and Lie derivative along a 1-form. Cotangent paths, cotangent homotopies, integration and invariance under homotopy. Connections, representations and parallel transport. Poisson fundamental group and groupoid.

Lecture 4. Stability of symplectic leaves: main theorem [2]

Poisson cohomology, restricted Poisson cohomology and relative Poisson cohomology. Coefficients. The Bott connection and linear Poisson holonomy. Precise statements of the stability theorems for symplectic leaves. Necessary versus sufficient conditions for stability. Examples.

Lecture 5. Lie algebroids: "Universal Stability Theorem" [1,2,3,6]

Leaves of a Lie algebroid. The Bott connection. The space of Lie algebroid structures on a fixed vector bundle. Precise statement of the "Universal Stability Theorem". Relationships with the stability theorems for foliations, group actions and Poisson structures. Examples.

Lecture 6. Proof of the stability theorems [2,8]

Proof of the (weak) Poisson stability theorem. Step1: Reduction to horizontally non-degenerate Poisson structures. Step 2: Translation to geometric triples. Step 3: symplectic leaves *vs* sections. Step 4: Linearizing around sections. Step 5: Setting up the functional. Step 6: Finding the critical points. The proofs of the other stability theorems.

Main References:

[1] A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lectures, vol. 10, American Math. Soc., Providence, 1999.

[2] M. Crainic and R.L. Fernandes, Stability of symplectic leaves, *preprint arXiv 0810.4437*.

[3] M. Crainic and I.Moerdjik, Deformations of Lie brackets: *J. Eur. Math. Soc.* **10** (2008), 1037--1059.

[4] R.L. Fernandes, Lie Algebroids, Holonomy and Characteristic Classes, *Adv. in Math*. **170**, (2002) 119-179.

[5] R. Langevin and H. Rosenberg, On stability of compact leaves and fibrations, *Topology* **16** (1977), 107-111.

[6] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Notes Series, 124. Cambridge University Press, Cambridge, 1987.

[7] D. Stowe, Stable orbits of differentiable group actions, *Trans. Amer. Math. Soc.* **277** (1983), 665-684.

[8] Y. Vorobjev, Coupling tensors and Poisson geometry near a single symplectic leaf, *Banach Center Publ.* **54**, (2001), 249-274.